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Theory of impedance networks: the two-point impedance and LC resonances

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Abstract

We present a formulation of the determination of the impedance between any two nodes in an impedance network. An impedance network is described by its Laplacian matrix $\mathbf{L}$ which has generally complex matrix elements. We show that by solving the equation $\mathbf{L}u_\alpha = \lambda_\alpha u_\alpha^*$ with orthonormal vectors u_a, the effective impedance between nodes p and q of the network is $Z_{pq} = \sum_\alpha (u_{\alpha p} - u_{\alpha q})^2/\lambda_\alpha$, where the summation is over all λ_α not identically equal to zero and $u_{\alpha p}$ is the pth component of u_α. For networks consisting of inductances L and capacitances C, the formulation leads to the occurrence of resonances at frequencies associated with the vanishing of λ_α. This curious result suggests the possibility of practical applications to resonant circuits. Our formulation is illustrated by explicit examples.

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1. Introduction

A classic problem in electric circuit theory that has attracted attention from Kirchhoff's time [1] to the present is the consideration of network resistances and impedances. While the evaluation of resistances and impedances can in principle be carried out for any given network using traditional, but often tedious, analysis such as the Kirchhoff's laws, there has been no conceptually simple solution. Indeed, the problem of computing the effective resistance between two arbitrary nodes in a resistor network has been studied by numerous authors (for a list of relevant references on resistor networks up to 2000 see, e.g., [2]). Particularly, an elementary exposition of the material can be found in Doyle and Snell [3].

However, past efforts prior to 2004 have been focused mainly on regular lattices and the use of Green's function technique, for which the analysis is most conveniently carried out when the network size is infinite [2, 4]. Little attention has been paid to *finite* networks, even though the latter are those occurring in applications. Furthermore, there has been very few studies on impedance networks. To be sure, studies have been carried out on electrical and optical properties of random impedance networks in binary composite media (for a review see [5]) and in dielectric resonances occurring in clusters embedded in a regular lattice [6]. But these are mostly approximate treatments on random media. More recently, Asad *et al* [7] evaluated the two-point capacitance in an infinite network of identical capacitances. When all impedances in a network are identical, however, Green's function technique used and the results are essentially the same as those of identical resistors.

In 2004 one of us proposed a new formulation of resistor networks which leads to an expression of the effective resistance between any two nodes in a network in terms of the eigenvalues and eigenvectors of the Laplacian matrix. Using this formulation one computes the effective resistance between two arbitrary nodes in any network which can be either finite or infinite [8]. This is a fundamentally new formulation. But the analysis presented in [8] makes use of the fact that for resistors the Laplacian matrix has real matrix elements. Consequently, the method does not extend to impedances whose Laplacian matrix elements are generally complex (see, e.g., [9]). In this paper we resolve this difficulty and extend the formulation of [8] to impedance networks.

Consider an impedance network $\mathcal{L}$ consisting of $\mathcal{N}$ nodes numbered $\alpha = 1, 2, \ldots, \mathcal{N}$. Let the impedance connecting nodes α and β be

$$z_{\alpha\beta} = z_{\beta\alpha} = r_{\alpha\beta} + \mathrm{i}x_{\alpha\beta}, \tag{1}$$

where $r_{\alpha\beta} = r_{\beta\alpha} \geqslant 0$ is the resistive part and $x_{\alpha\beta} = x_{\beta\alpha}$ is the reactive part, which is positive for inductances and negative for capacitances. Here, $\mathrm{i} = \sqrt{-1}$ often denoted by $\mathrm{j} = \sqrt{-1}$ in alternating current (ac) circuit theory [9]. In this paper we shall use i and j interchangeably. The admittance y connecting two nodes is the reciprocal of the impedance. For example, $y_{\alpha\beta} = y_{\beta\alpha} = 1/z_{\alpha\beta}$.

Denote the electric potential at node α by V_α and the *net* current flowing into the network (from the outside world) at node α by I_α. Both V_α and I_α are generally complex in the *phasor* notation used in ac circuit theory [9]. Since there is neither source nor sink of currents, one has the conservation rule

$$\sum_{\alpha=1}^{\mathcal{N}} I_\alpha = 0. \tag{2}$$

The Kirchhoff equation for the network reads

$$\mathbf{L}\vec{V} = \vec{I}, \tag{3}$$

where

$$\mathbf{L} = \begin{pmatrix} y_1 & -y_{12} & \ldots & -y_{1\mathcal{N}} \\ -y_{21} & y_2 & \ldots & -y_{2\mathcal{N}} \\ \vdots & \vdots & \ddots & \vdots \\ -y_{\mathcal{N}1} & -y_{\mathcal{N}2} & \ldots & y_{\mathcal{N}} \end{pmatrix}, \tag{4}$$

with

$$y_\alpha \equiv \sum_{\beta=1(\beta\neq\alpha)}^{\mathcal{N}} y_{\alpha\beta}, \tag{5}$$

is the Laplacian matrix associated with the network $\mathcal{L}$. In (3), $\vec{V}$ and $\vec{I}$ are $\mathcal{N}$-vectors whose components are respectively V_α and I_α.

Here, we need to solve (3) for $\vec{V}$ for a given current configuration $\vec{I}$. The effective impedance between nodes p and q, the quantity we wish to compute, is by definition the ratio

$$Z_{pq} = \frac{V_p - V_q}{I}, \tag{6}$$

where V_p and V_q are solved from (3) with

$$I_\alpha = I(\delta_{\alpha p} - \delta_{\alpha q}). \tag{7}$$

The crux of the matter is to solve the Kirchhoff equation (3) for $\vec{I}$ given by (7). The difficulty lies in the fact that, since the matrix $\mathbf{L}$ is singular, equation (3) cannot be formally inverted.

To circumvent this difficulty we proceed as in [8] to consider instead the equation

$$\mathbf{L}(\epsilon)\vec{V}(\epsilon) = \vec{I}, \tag{8}$$

where

$$\mathbf{L}(\epsilon) = \mathbf{L} + \epsilon\mathbf{I}, \tag{9}$$

and $\mathbf{I}$ is the identity matrix. The matrix $\mathbf{L}(\epsilon)$ now has an inverse and we can proceed by applying the arsenal of linear algebra. We take the $\epsilon \to 0$ limit at the end and do not expect any problem since we know there is a physical solution.

The crucial step is the computation of the inverse matrix $\mathbf{L}^{-1}(\epsilon)$. For this purpose it is useful to first recall the approach for resistor networks.

In the case of resistor networks the matrix $\mathbf{L}(\epsilon)$ is real symmetric and hence it has orthonormal eigenvectors $\psi_\alpha(\epsilon)$ with eigenvalues $\lambda_\alpha(\epsilon) = \lambda_\alpha + \epsilon$ determined from the eigenvalue equation

$$\mathbf{L}(\epsilon)\psi_\alpha(\epsilon) = \lambda_\alpha(\epsilon)\psi_\alpha(\epsilon), \qquad i = 1, 2, \ldots, \mathcal{N}. \tag{10}$$

Now a real Hermitian matrix $\mathbf{L}(\epsilon)$ is diagonalized by the unitary transformation $\mathbf{U}^\dagger(\epsilon)\mathbf{L}(\epsilon)\mathbf{U}(\epsilon) = \Lambda(\epsilon)$, where $\mathbf{U}(\epsilon)$ is a unitary matrix whose columns are the orthonormal eigenvectors $\psi_\alpha(\epsilon)$ and $\Lambda(\epsilon)$ is a diagonal matrix with diagonal elements $\lambda_\alpha(\epsilon) = \lambda_\alpha + \epsilon$. The inverse of this relation leads to $\mathbf{L}^{-1}(\epsilon) = \mathbf{U}(\epsilon)\Lambda^{-1}(\epsilon)\mathbf{U}^\dagger(\epsilon)$.[3] In this way we find the effective resistance between nodes p and q to be [8]

$$R_{pq} = \sum_{\alpha=2}^{\mathcal{N}} \frac{1}{\lambda_\alpha}|\psi_{\alpha p} - \psi_{\alpha q}|^2, \tag{11}$$

where the summation is over all nonzero eigenvalues, and $\psi_{\alpha p}$ is the pth component of $\psi_\alpha(0)$. Here the $\alpha = 1$ term in the summation with $\lambda_1(\epsilon) = \epsilon$ and $\psi_{1p}(\epsilon) = 1/\sqrt{\mathcal{N}}$ drops out (before taking the $\epsilon \to 0$ limit) due to the conservation rule (2). It can be shown that there is no other zero eigenvalue if the network is singly connected. Relation (11) is the main result of [8].

2. Impedance networks

For impedance networks the Laplacian matrix $\mathbf{L}$ is symmetric and generally complex and thus

$$\mathbf{L}^\dagger = \mathbf{L}^* \neq \mathbf{L},$$

where $*$ denotes the complex conjugation and † denotes the hermitian conjugate. Therefore $\mathbf{L}$ is not Hermitian and cannot be diagonalized as described in the preceding section.

However, the matrix $\mathbf{L}^\dagger\mathbf{L}$ is always Hermitian and has non-negative eigenvalues. Write the eigenvalue equation as

$$\mathbf{L}^\dagger\mathbf{L}\psi_\alpha = \sigma_\alpha\psi_\alpha, \qquad \sigma_\alpha \geqslant 0, \qquad \alpha = 1, 2, \ldots, \mathcal{N}. \tag{12}$$

One verifies that one eigenvalue is $\sigma_1 = 0$ with $\psi_1 = \{1, 1, \ldots, 1\}^T/\sqrt{\mathcal{N}}$, where the superscript T denotes the transpose. For complex $\mathbf{L}$ there can exist other zero eigenvalues (see below).

[3] The equivilent of the method we use in obtaining (11) is known in mathematics literature as the pseudo-inverse method (see, e.g., [10, 11]).

To facilitate considerations, we again introduce $\mathbf{L}(\epsilon)$ as in (9) and rewrite (12) as

$$\mathbf{L}^{\dagger}(\epsilon)\mathbf{L}(\epsilon)\psi_{\alpha}(\epsilon) = \sigma_{\alpha}(\epsilon)\psi_{\alpha}(\epsilon), \qquad \sigma_{\alpha}(\epsilon) \geqslant 0, \quad \alpha = 1, 2, \ldots, \mathcal{N}, \tag{13}$$

where ϵ is small. Now one eigenvalue is $\sigma_1(\epsilon) = \epsilon^2$ with $\psi_1(\epsilon) = \{1, 1, \ldots, 1\}^T/\sqrt{\mathcal{N}}$. For other eigenvectors we make use of the theorem established in the following section (see also [12]) that there exist $\mathcal{N}$ orthonormal vectors $u_{\alpha}(\epsilon)$ satisfying the equation

$$\mathbf{L}(\epsilon)u_{\alpha}(\epsilon) = \lambda_{\alpha}(\epsilon)u_{\alpha}^{*}(\epsilon), \qquad a = 1, 2, \ldots, \mathcal{N}, \tag{14}$$

where

$$\lambda_{\alpha}(\epsilon) = \sqrt{\sigma_{\alpha}(\epsilon)}\,\mathrm{e}^{\mathrm{i}\theta_{\alpha}(\epsilon)}, \qquad \theta_{\alpha}(\epsilon) = \text{real}. \tag{15}$$

Particularly, we can take

$$\lambda_1(\epsilon) = \sqrt{\sigma_1(\epsilon)} = \epsilon, \qquad \theta_1(\epsilon) = 0. \tag{16}$$

Equation (14) plays the role of the eigenvalue equation (10) for resistors.

We next construct a unitary matrix $\mathbf{U}(\epsilon)$ whose columns are $u_{\alpha}(\epsilon)$. Using (14) and the fact that $\mathbf{L}(\epsilon)$ is symmetric, one verifies that $\mathbf{L}(\epsilon)$ is diagonalized by the transformation

$$\mathbf{U}^{T}(\epsilon)\mathbf{L}(\epsilon)\mathbf{U}(\epsilon) = \Delta(\epsilon),$$

where $\Delta(\epsilon)$ is a diagonal matrix with diagonal elements $\lambda_{\alpha}(\epsilon)$. The inverse of this relation leads to

$$\mathbf{L}^{-1}(\epsilon) = \mathbf{U}(\epsilon)\Delta^{-1}(\epsilon)\mathbf{U}^{T}(\epsilon), \tag{17}$$

where $\Delta^{-1}(\epsilon)$ is a diagonal matrix with diagonal elements $1/\lambda_{\alpha}(\epsilon)$. We can now use (17) to solve (8) to obtain, after using (6),

$$Z_{pq} = \lim_{\epsilon\to 0} \sum_{\alpha=1}^{\mathcal{N}} \frac{1}{\lambda_{\alpha}(\epsilon)}(u_{\alpha p}(\epsilon) - u_{\alpha q}(\epsilon))^2, \tag{18}$$

where $u_{\alpha p}$ is the pth component of the orthonormal vector $u_{\alpha}(\epsilon)$.

Now the term $\alpha = 1$ in the summation drops out before taking the limit just like in the case of resistors [8] since $\lambda_1(\epsilon) = \epsilon$ and $u_{1p}(\epsilon) = u_{1q}(\epsilon) =$ constant. If there exist other eigenvalues $\lambda_{\alpha}(\epsilon) = \epsilon$ with $u_{\alpha p}(\epsilon) \neq$ constant, a situation which can occur when there are pure reactances L and C, the corresponding terms in (18) diverge in the $\epsilon \to 0$ limit at specific frequencies ω in an ac circuit. Then one obtains the effective impedance

$$\begin{aligned} Z_{pq} &= \sum_{\alpha=2}^{\mathcal{N}} \frac{1}{\lambda_{\alpha}}(u_{\alpha p} - u_{\alpha q})^2, &&\text{if} && \lambda_{\alpha} \neq 0, \quad \alpha \geqslant 2 \\ &= \infty, &&\text{if there exists} && \lambda_{\alpha} = 0, \quad \alpha \geqslant 2. \end{aligned} \tag{19}$$

Here $u_{\alpha p} = u_{\alpha p}(0)$. The physical interpretation of $Z = \infty$ is the occurrence of a *resonance* in an ac circuit at frequencies where $\lambda_{\alpha} = 0$, meaning it requires essentially a zero input current I to maintain potential differences at these frequencies.

Expression (19) is our main result for impedance networks.

In the case of pure resistors, the Laplacian $\mathbf{L}(\epsilon)$ and the eigenvalues $\lambda_{\alpha}(\epsilon)$ in (10) are real, so without loss of generality we can take $\psi_{\alpha}(\epsilon)$ to be real (see example 3 in section 5), and use $u_{\alpha}(\epsilon) = \psi_{\alpha}(\epsilon)$ in (14) with $\theta_{\alpha}(\epsilon) = 0$. Then $u_{\alpha p}(\epsilon)$ in (18) is real and (19) coincides with (11) for resistors. There is no $\lambda_{\alpha} = 0$ other than $\lambda_1 = 0$, and there is no resonance.

3. Complex symmetric matrix

For completeness in this section we give a proof of the theorem which asserts (14) and determines u_α for a complex symmetric matrix. Our proof parallels that in [12].

Theorem. *Let* $\mathbf{L}$ *be an* $n \times n$ *symmetric matrix with generally complex elements. Write the eigenvalue equation of* $\mathbf{L}^\dagger\mathbf{L}$ *as*

$$\mathbf{L}^\dagger\mathbf{L}\psi_\alpha = \sigma_\alpha\psi_\alpha, \qquad \sigma_\alpha \geqslant 0, \quad \alpha = 1, 2, \ldots, n. \tag{20}$$

Then, there exist n orthonormal vectors u_α *satisfying the relation*

$$\mathbf{L}u_\alpha = \lambda_\alpha u_\alpha^*, \qquad \alpha = 1, 2, \ldots, n, \tag{21}$$

where $*$ *denotes the complex conjugation and* $\lambda_\alpha = \sqrt{\sigma_\alpha}\,\mathrm{e}^{\mathrm{i}\theta_\alpha}, \theta_\alpha = real$.

For nondegenerate σ_α *we can take* $u_\alpha = \psi_\alpha$*; for degenerate* σ_α*, the u's are linear combinations of the degenerate* ψ_α*. In either case the phase factor* θ_α *of* λ_α *is determined by applying (21).*

Remark

1. The λ_α's are the eigenvalues of $\mathbf{L}$ if u_α's are real.
2. If $\{u_\alpha, \lambda_\alpha\}$ is a solution of (21), then $\{u_\alpha\,\mathrm{e}^{\mathrm{i}\tau}, \lambda_\alpha\,\mathrm{e}^{2\mathrm{i}\tau}\}$, $\tau =$ real, is also a solution of (21).
3. While the procedure of constructing u_α in the degenerate case appears to be involved, as demonstrated in examples given in section 5 the orthonormal u's can often be determined quite directly in practice.
4. If $\mathbf{L}$ is real, then as aforementioned it has real eigenvalues and eigenvectors, and we can take these real eigenvectors to be u_α in (21) with λ_α real non-negative.

Proof. Since $\mathbf{L}^\dagger\mathbf{L}$ is Hermitian its nondegenerate eigenvectors ψ_α can be chosen to be orthonormal. For the eigenvector ψ_α with nondegenerate eigenvalue σ_α, construct a vector

$$\phi_\alpha = (\mathbf{L}\psi_\alpha)^* + c_\alpha\psi_\alpha, \tag{22}$$

where c_α is any complex number. It is readily verified that we have

$$\mathbf{L}^\dagger\mathbf{L}\phi_\alpha = \sigma_\alpha\phi_\alpha, \tag{23}$$

so ϕ_α is also an eigenvector of $\mathbf{L}^\dagger\mathbf{L}$ with the same eigenvalue σ_α. It follows that if σ_α is nondegenerate then ϕ_α and ψ_α must be proportional, namely,

$$\mathbf{L}\psi_\alpha = \lambda_\alpha\psi_\alpha^* \tag{24}$$

for some λ_α. The substitution of (24) into (23) with ϕ_α given by (22) now yields $|\lambda_\alpha|^2 = \sigma_\alpha$ or $\lambda_\alpha = \sqrt{\sigma_\alpha}\,\mathrm{e}^{\mathrm{i}\theta_\alpha}$. Thus, for nondegenerate σ_α we simply choose $u_\alpha = \psi_\alpha$ and use (21) and (20) to determine the phase factor θ_α. This establishes the theorem for nondegenerate λ_α.

For degenerate eigenvalues of $\mathbf{L}^\dagger\mathbf{L}$, say, $\sigma_1 = \sigma_2 = \sigma$ with linearly independent eigenvectors ψ_1 and ψ_2, we construct

$$v_1 = (\mathbf{L}\psi_1)^* + \sqrt{\sigma}\,\mathrm{e}^{\mathrm{i}\theta_1}\psi_1 \qquad v_2 = (\mathbf{L}\psi_2)^* + \sqrt{\sigma}\,\mathrm{e}^{\mathrm{i}\theta_2}\psi_2, \tag{25}$$

where the choice of the real phase factors θ_1, θ_2 is at our disposal. We choose θ_1, θ_2 to make v_1 and v_2 linearly independent to satisfy

$$\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)} = (v_2, v_1)^*/(v_2, v_1), \tag{26}$$

where $(y, z) = (y^T)^*z$ is the inner product of vectors y and z.

Now one has

$$\begin{aligned} \mathbf{L}v_1 &= \sqrt{\sigma}\,\mathrm{e}^{\mathrm{i}\theta_1}v_1^*, \qquad & \mathbf{L}v_2 &= \sqrt{\sigma}\,\mathrm{e}^{\mathrm{i}\theta_2}v_2^* \\ \mathbf{L}^\dagger\mathbf{L}v_1 &= \sigma v_1, \qquad & \mathbf{L}^\dagger\mathbf{L}v_2 &= \sigma v_2. \end{aligned} \tag{27}$$

Write

$$u_1 = v_1/|v_1|, \tag{28}$$

where $|v| = \sqrt{(v, v)}$ is the norm of v, and construct $y = v_2 - (v_2, u_1)u_1$ which is orthogonal to u_1. Next write

$$u_2 = y/|y|. \tag{29}$$

Then, it can be verified by using (26) that u_1 and u_2 are orthonormal and satisfy

$$\begin{aligned} \mathbf{L}u_1 &= \sqrt{\sigma}\, \mathrm{e}^{\mathrm{i}\theta_1} u_1^* \\ \mathbf{L}u_2 &= \sqrt{\sigma}\, \mathrm{e}^{\mathrm{i}\theta_2} u_2^*. \end{aligned} \tag{30}$$

In addition, both u_1 and u_2 are eigenvectors of $\mathbf{L}^\dagger\mathbf{L}$ with the same eigenvalue σ, hence are orthogonal to ψ_α, $\alpha \geqslant 3$. This establishes the theorem. □

In the case of multi-degeneracy, a similar analysis can be carried out by starting from a set of v_α to construct u_α's by using, say, the Gram–Schmidt orthonormalization procedure. For details we refer to [13].

4. Resonances

If there exist eigenvalues $\lambda_\alpha = 0$, $\alpha \geqslant 2$, a situation which can occur at specific frequencies ω in an ac circuit, then the effective impedance (19) between *any* two nodes diverges and the network is in resonance.

In an ac circuit resonances occur when the impedances are pure reactances (capacitances or inductances). The simplest example of a resonance is a circuit containing two nodes connecting an inductance L and capacitance C in parallel. It is well known that this LC circuit is resonant with an external ac source at the frequency $\omega = 1/\sqrt{LC}$. This is most simply seen by noting that the two nodes are connected by an admittance $y_{12} = \mathrm{j}\omega C + 1/\mathrm{j}\omega L = \mathrm{j}(\omega C - 1/\omega L)$, and hence $Z_{12} = 1/y_{12}$ diverges at $\omega = 1/\sqrt{LC}$.

Alternately, using our formulation, the Laplacian matrix is

$$\mathbf{L} = y_{12}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{31}$$

so that $\mathbf{L}^*\mathbf{L}$ has eigenvalues $\sigma_1 = 0$, $\sigma_2 = 4|y_{12}|^2$ and we have $\lambda_1 = 0$ as expected. In addition, we also have $\lambda_2 = 0$ when $y_{12} = 0$ at the frequency $\omega = 1/\sqrt{LC}$. This is the occurrence of a resonance.

An extension of this consideration to N reactances in a ring is discussed in example 2 in the following section.

5. Examples

Example 1. A numerical example

Examples of applications of the formulation (19) are given in this section. It is instructive to work out a numerical example as an illustration.

Consider three impedances $z_{12} = \mathrm{i}\sqrt{3}$, $z_{23} = -\mathrm{i}\sqrt{3}$, $z_{31} = 1$ connected in a ring as shown in figure 1 where $\mathrm{i} = \mathrm{j} = \sqrt{-1}$. We have the Laplacian

$$\mathbf{L} = \begin{pmatrix} 1 - \mathrm{i}/\sqrt{3} & \mathrm{i}/\sqrt{3} & -1 \\ \mathrm{i}/\sqrt{3} & 0 & -\mathrm{i}/\sqrt{3} \\ -1 & -\mathrm{i}/\sqrt{3} & 1 + \mathrm{i}/\sqrt{3} \end{pmatrix}. \tag{32}$$

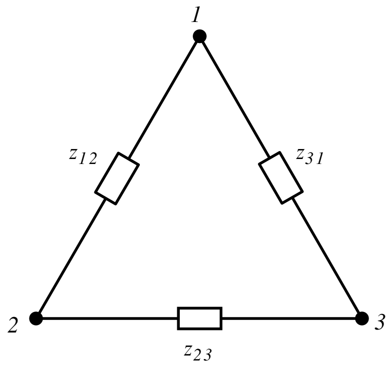

Figure 1. An example of three impedances in a ring.

Substituting **L** into (12) we find the following nondegenerate eigenvalues and orthonormal eigenvectors of $\mathbf{L}^\dagger\mathbf{L}$,

$$\begin{aligned}
\sigma_1 &= 0, & \psi_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \\
\sigma_2 &= 3 - 2\sqrt{2}, & \psi_2 &= \frac{1}{\sqrt{24 - 6\sqrt{2}}} \begin{pmatrix} 2 - \sqrt{2} + \mathrm{i}\sqrt{3} \\ -\sqrt{2} - 1 - \mathrm{i}\sqrt{3} \\ 2\sqrt{2} - 1 \end{pmatrix}, \\
\sigma_3 &= 3 + 2\sqrt{2}, & \psi_3 &= \frac{1}{\sqrt{24 + 6\sqrt{2}}} \begin{pmatrix} 2 + \sqrt{2} + \mathrm{i}\sqrt{3} \\ \sqrt{2} - 1 - \mathrm{i}\sqrt{3} \\ -2\sqrt{2} - 1 \end{pmatrix}.
\end{aligned} \tag{33}$$

Since the eigenvalues are nondegenerate, according to the theorem we take $u_i = \psi_i, i = 1, 2, 3$. Using these expressions we obtain from (21)

$$\begin{aligned}
\sqrt{\sigma_2} &= \sqrt{2} - 1, & \mathrm{e}^{\mathrm{i}\theta_2} &= \tfrac{1}{7}[3\sqrt{2} - 2 + \mathrm{i}\sqrt{3}(2\sqrt{2} + 1)] \\
\sqrt{\sigma_3} &= \sqrt{2} + 1, & \mathrm{e}^{\mathrm{i}\theta_3} &= \tfrac{1}{7}[3\sqrt{2} + 2 + \mathrm{i}\sqrt{3}(2\sqrt{2} - 1)].
\end{aligned} \tag{34}$$

Now (19) reads

$$Z_{pq} = \frac{\mathrm{e}^{-\mathrm{i}\theta_2}}{\sqrt{\sigma_2}}(u_{2p} - u_{2q})^2 + \frac{\mathrm{e}^{-\mathrm{i}\theta_3}}{\sqrt{\sigma_3}}(u_{3p} - u_{3q})^2, \tag{35}$$

using which one obtains the impedances

$$Z_{12} = 3 + \mathrm{i}\sqrt{3}, \qquad Z_{23} = 3 - \mathrm{i}\sqrt{3}, \qquad Z_{31} = 0. \tag{36}$$

These values agree with results of direct calculation using Ohm's law.

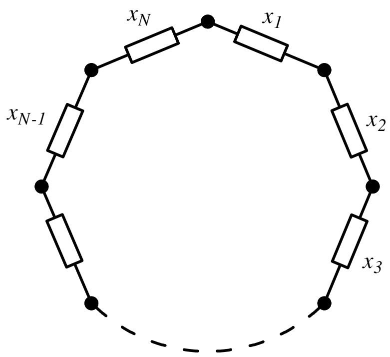

Figure 2. A ring of N reactances.

Example 2. Resonance in a one-dimensional ring of N reactances

Consider N reactances $jx_1, jx_2, \ldots, jx_N$ connected in a ring as shown in figure 2, where $x = \omega L$ for inductance L and $x = -1/\omega C$ for capacitance C at ac frequency ω. The Laplacian assumes the form

$$\mathbf{L} = \frac{1}{j}\begin{pmatrix} y_1+y_N & -y_1 & 0 & \cdots & 0 & 0 & -y_N \\ -y_1 & y_1+y_2 & -y_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -y_{N-1} & y_{N-1}+y_N & -y_N \\ -y_N & 0 & 0 & \cdots & 0 & -y_1 & y_N+y_1 \end{pmatrix} \tag{37}$$

where $y_i = 1/x_i$. The Laplacian $\mathbf{L}$ has one zero eigenvalue $\lambda_1 = 0$ as aforementioned. The product of the other $N-1$ eigenvalues λ_α of $\mathbf{L}$ is known from graph theory [14, 15] to be equal to N times its spanning tree generating function with edge weights $y_1, y_2, \ldots, y_N$. Now the N spanning trees are easily written down and as a result we obtain

$$\begin{aligned}\prod_{i=2}^{N}\lambda_\alpha &= N(-\mathrm{j})^{N-1}\left(\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_N}\right)y_1y_2\cdots y_N \\ &= N(-\mathrm{j})^{N-1}(x_1+x_2+\cdots+x_N)/x_1x_2\cdots x_N.\end{aligned} \tag{38}$$

It follows that there exists another zero eigenvalue, and hence a resonance, if $x_1 + x_2 + \cdots + x_N = 0$. This determines the resonance frequency ω.

Example 3. A one-dimensional ring of N equal impedances

In this example we consider N equal impedances z connected in a ring. We have

$$\mathbf{L} = y\mathbf{T}_N^{\mathrm{per}}, \qquad \mathbf{L}^\dagger = y^*\mathbf{T}_N^{\mathrm{per}}, \qquad \mathbf{L}^\dagger\mathbf{L} = |y|^2\left(\mathbf{T}_N^{\mathrm{per}}\right)^2, \tag{39}$$

where $y = 1/z$ and

$$\mathbf{T}_N^{\mathrm{per}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{40}$$

Thus $\mathbf{L}$ and $\mathbf{L}^\dagger\mathbf{L}$ all have the same eigenvectors. The eigenvalues and orthonormal eigenvectors of $\mathbf{T}_N^{\text{per}}$ are

$$\mu_n = 2[1-\cos(2n\pi/N)] = 4\cos^2(n\pi/N)$$
$$\psi_n = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \\ \vdots \\ \omega^{(N-1)n}\end{pmatrix}, \qquad n = 0, 1, \ldots, N-1, \tag{41}$$

where $\omega = \mathrm{e}^{\mathrm{i}2\pi/N}$. The eigenvalues of $\mathbf{L}^\dagger\mathbf{L}$ are

$$\sigma_n = |y|^2\mu_n^2. \tag{42}$$

Since

$$\sigma_{N-n} = \sigma_n, \tag{43}$$

the corresponding eigenvectors are degenerate and we need to construct vectors u_{n1} and u_{n2} for $0 < n < N/2$. For $N =$ even, however, the eigenvalue $\sigma_{N/2}$ is non-degenerate and needs to be considered separately.

For $0 < n < N/2$ the degenerate eigenvectors

$$\psi_n \qquad \text{and} \qquad \psi_{N-n} = \psi_n^* \tag{44}$$

are not orthonormal. Then we construct linear combinations

$$u_{n1} = \frac{\psi_n + \psi_n^*}{\sqrt{2}} = \sqrt{\frac{2}{N}}\begin{pmatrix} 1 \\ \cos\frac{2n\pi}{N} \\ \cos\frac{4n\pi}{N} \\ \vdots \\ \cos\frac{2(N-1)n\pi}{N}\end{pmatrix},$$
$$u_{n2} = \frac{\psi_n - \psi_n^*}{\sqrt{2}i} = \sqrt{\frac{2}{N}}\begin{pmatrix} 0 \\ \sin\frac{2n\pi}{N} \\ \sin\frac{4n\pi}{N} \\ \vdots \\ \sin\frac{2(N-1)n\pi}{N}\end{pmatrix}, \qquad n = 1, 2, \ldots, \left[\frac{N-1}{2}\right], \tag{45}$$

which are orthonormal, where $[x]$ is the integral part of x. The u's are eigenvectors of $\mathbf{L}^\dagger\mathbf{L}$ with the same eigenvalue $\sigma_n = |y|^2\mu_n^2$. For $N =$ even we have an additional non-degenerate eigenvector

$$u_{N/2} = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ -1\end{pmatrix}. \tag{46}$$

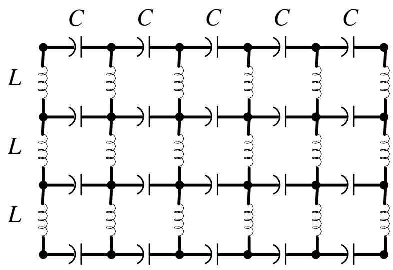

Figure 3. A 6 × 4 network of capacitances C and inductances L.

We next use (21) to determine the phase factors θ_{n1} and θ_{n2}. Comparing the eigenvalue equation

$$\begin{aligned} &\mathbf{L}u_{n1} = (y\mu_n)u_{n1} && \text{with} && \mathbf{L}u_{n1} = (|y|\mu_n)\,\mathrm{e}^{\mathrm{i}\theta_{n1}}u_{n1}^*, \\ &\mathbf{L}u_{n2} = (y\mu_n)u_{n2} && \text{with} && \mathbf{L}u_{n2} = (|y|\mu_n)\,\mathrm{e}^{\mathrm{i}\theta_{n2}}u_{n2}^*, \quad \text{and} \\ &\mathbf{L}u_{N/2} = 4(y)u_{N/2} && \text{with} && \mathbf{L}u_{N/2} = 4|y|\,\mathrm{e}^{\mathrm{i}\theta_{N/2}}u_{N/2}^*, \end{aligned} \tag{47}$$

we obtain

$$\theta_{n1} = \theta_{n2} = \theta_{N/2} = \theta, \tag{48}$$

where θ is given by $y = |y|\,\mathrm{e}^{\mathrm{i}\theta}$.

We now use (19) to compute the impedance between nodes p and q to obtain

$$Z_{pq} = \frac{2}{Ny}\sum_{n=1}^{[\frac{N-1}{2}]}\frac{1}{\mu_n}\left[\left(\cos\frac{2np\pi}{N} - \cos\frac{2nq\pi}{N}\right)^2 - \mathrm{i}^2\left(\sin\frac{2np\pi}{N} - \sin\frac{2nq\pi}{N}\right)^2\right] + E, \tag{49}$$

where $[x]$ denotes the integral part of x and

$$\begin{aligned} E &= \frac{1}{2Ny}\left[(-1)^p - (-1)^q\right]^2, && N = \text{even} \\ &= 0, && N = \text{odd}. \end{aligned} \tag{50}$$

After some manipulation it is reduced to

$$Z_{pq} = \frac{z}{N}\sum_{n=1}^{N-1}\frac{|\mathrm{e}^{\mathrm{i}2np\pi/N} - \mathrm{e}^{\mathrm{i}2nq\pi/N}|^2}{2[1-\cos(2n\pi/N)]}. \tag{51}$$

This expression has been evaluated in [8] with the result

$$Z_{pq} = z|p-q|\left[1 - \frac{|p-q|}{N}\right], \tag{52}$$

which is the expected impedance of two impedances $|p-q|z$ and $(N-|p-q|)z$ connected in parallel as in a ring. This completes the evaluation of Z_{pq}.

Example 4. Networks of inductances and capacitances

As an example of networks of inductances and capacitances, we consider an $M \times N$ array of nodes forming a rectangular net with free boundaries as shown in figure 3. The nodes are connected by capacitances C in the M directions and inductances L in the N direction.

The Laplacian of the network is

$$\mathbf{L} = (\mathrm{j}\omega C)\,\mathbf{T}_M^{\mathrm{free}} \otimes \mathrm{I}_N - \left(\frac{\mathrm{j}}{\omega L}\right) \mathrm{I}_M \otimes \mathbf{T}_N^{\mathrm{free}}, \tag{53}$$

where $\mathbf{T}_M^{\mathrm{free}}$ is the $M \times M$ matrix

$$\mathbf{T}_M^{\mathrm{free}} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}, \tag{54}$$

and $\mathbf{I}_N$ is the $N \times N$ identity matrix. This gives

$$\mathbf{L}^*\mathbf{L} = (\omega C)^2 \mathbf{U}_M^{\mathrm{free}} \otimes \mathbf{I}_N - 2\left(\frac{C}{L}\right) \mathbf{T}_M^{\mathrm{free}} \otimes \mathbf{T}_N^{\mathrm{free}} + \left(\frac{1}{\omega L}\right)^2 \mathbf{I}_M \otimes \mathbf{U}_N^{\mathrm{free}}, \tag{55}$$

where $\mathbf{U}_M^{\mathrm{free}}$ is the $M \times M$ matrix

$$\mathbf{U}_M^{\mathrm{free}} = \begin{pmatrix} 2 & -3 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -3 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -3 & 2 \end{pmatrix}. \tag{56}$$

Now $\mathbf{T}_M^{\mathrm{free}}$ has eigenvalues

$$\lambda_m = 2(1 - \cos\theta_m) = 4\sin^2(\theta_m/2), \qquad \theta_m = \frac{m\pi}{M} \tag{57}$$

and eigenvector $\psi_m^{(M)}$ whose components are

$$\psi_{mx}^{(M)} = \begin{cases} \frac{1}{\sqrt{M}}, & m = 0, \text{ for all } x, \\ \sqrt{\frac{2}{M}}\cos\left(x + \frac{1}{2}\right)\theta_m, & m = 1, 2, \ldots, M-1, \text{ for all } x. \end{cases} \tag{58}$$

It follows that $\mathbf{L}^*\mathbf{L}$ has eigenvectors

$$\psi_{(m,n);(x,y)}^{\mathrm{free}} = \psi_{mx}^{(M)} \psi_{ny}^{(N)} \tag{59}$$

and eigenvalues

$$\sigma_{mn} = 16\left(\omega C \sin^2\frac{\theta_m}{2} - \frac{1}{\omega L}\sin^2\frac{\phi_n}{2}\right)^2, \tag{60}$$

where $\theta_m = m\pi/M$, $\phi_n = n\pi/N$. This gives

$$\begin{aligned} \lambda_{mn} &= 4\mathrm{j}\left[\omega C \sin^2(\theta_m/2) - \frac{1}{\omega L}\sin^2(\phi_n/2)\right] \\ &= \sqrt{\sigma_{mn}}\,\mathrm{e}^{\mathrm{i}\theta_{mn}}, \qquad \theta_{mn} = \pm\pi/2. \end{aligned} \tag{61}$$

Since the vectors $\psi_{(m,n);(x,y)}^{\mathrm{free}}$ are orthonormal and non-degenerate, according to the theorem we can use these vectors in (19) to obtain the impedance between nodes (x_1, y_1) and (x_2, y_2).

This gives

$$
\begin{aligned}
Z^{\mathrm{free}}_{(x_1,y_1);(x_2,y_2)} &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}{}_{(m,n)\neq(0,0)} \frac{\left(\psi^{\mathrm{free}}_{(m,n);(x_1,y_1)} - \psi^{\mathrm{free}}_{(m,n);(x_2,y_2)}\right)^2}{\lambda_{mn}} \\
&= \frac{-\mathrm{j}}{N\omega C}|x_1-x_2| + \frac{\mathrm{j}\omega L}{M}|y_1-y_2| + \frac{2\mathrm{j}}{MN} \\
&\quad\times \sum_{m=1}^{M-1}\sum_{n=1}^{N-1} \frac{\left[\cos\left(x_1+\frac{1}{2}\right)\theta_m \cos\left(y_1+\frac{1}{2}\right)\phi_n - \cos\left(x_2+\frac{1}{2}\right)\theta_m \cos\left(y_2+\frac{1}{2}\right)\phi_n\right]^2}{-\omega C(1-\cos\theta_m) + \frac{1}{\omega L}(1-\cos\phi_n)}.
\end{aligned}
\tag{62}
$$

As discussed in section 4, resonances occur at ac frequencies determined from $\lambda_{mn} = 0$. Thus, there are $(M-1)(N-1)$ distinct resonance frequencies given by

$$
\omega_{mn} = \left|\frac{\sin(n\pi/2N)}{\sin(m\pi/2M)}\right| \frac{1}{\sqrt{LC}}, \qquad m = 1,\ldots,M-1; \quad n = 1,\ldots,N-1. \tag{63}
$$

A similar result can be found for an $M \times N$ net with toroidal boundary conditions. However, due to the degeneracy of eigenvalues, in that case there are $[(M+1)/2][(N+1)/2]$ distinct resonance frequencies, where $[x]$ is the integral part of x. It is of pertinent interest to note that a network can become resonant at a spectrum of distinct frequencies, and these resonances occur in the effective impedances between *any* two nodes.

In the limit of $M, N \to \infty$, (63) becomes continuous indicating that the network is resonant at all frequencies. This is verified by replacing the summations by integrals in (62) to yield the effective impedance between two nodes (x_1, y_1) and (x_2, y_2),

$$
Z^{\infty}_{(x_1,y_1);(x_2,y_2)} = \frac{\mathrm{j}}{4\pi^2}\int_0^{2\pi}\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi \left[\frac{1-\cos[(x_1-x_2)\theta]\cos[(y_1-y_2)\phi]}{-\omega C(1-\cos\theta) + \frac{1}{\omega L}(1-\cos\phi)}\right], \tag{64}
$$

which diverges logarithmically[4].

6. Summary

We have presented a formulation of impedance networks which permits the evaluation of the effective impedance between arbitrary two nodes. The resulting expression is (19) where u_α and λ_a are those given in (14). In the case of reactance networks, our analysis indicates that resonances occur at ac frequencies ω determined by the vanishing of λ_a. This curious result suggests the possibility of practical applications of our formulation to resonant circuits.

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[4] Detailed steps leading to (62) and (64) can be found in equations (37) and (40) of [8].

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